**Thinking About Exponents**

**Part 1**

Many adults dream about a day in the future when they can retire from working their job and spend more time on their hobbies, traveling, and spending time with friends and family. To realize this dream, people have to save money so that they can afford to do all the things that they want to do!

Suppose a 35-year-old person was able to save $10,000. They plan to invest this money into a retirement savings account that earns 12% annual interest (by the way, we haven’t seen such a high rate of return since 1989!). This would mean that their money would double in value every 6 years.

1. Based on this investment, complete the table.

|  |  |
| --- | --- |
| Years since investment | Amount of money in the investment |
| 0 | $10,000 |
| 6 | $20,000 |
| 12 | $40,000 |
| 18 | $80,000 |
| 24 | $160,000 |
| 30 | $320,000 |
| 36 | $640,000 |

2. Suppose *n* represents the number of 6-year periods that have passed. For example, *n* = 2 means 12 years have passed or *n* = 5 means 30 years have passed. Explain what each of the following expressions mean in the context of this investment situation.

a)

Since *n* = 0, this represents the initial investment of $10,000.

b)

Since *n* = 1, this represents the account balance after 6 years.

c)

Since *n* = 3, this represents the account balance after 18 years.

d)

Since *n* = 6, this represents the account balance after 36 years.

3. Write a formula that could be used to find the amount of money, *A*, in the investment for any value of *n* where *n* still represents the number of 6 year periods that have passed since the initial investment.

4. Use your formula in #3 to determine how much money will be in the investment if the person wants to retire at age 56.

Since 21 years will have passed when this person reaches age 56, . That is, 3.5 six-year periods will have passed when the person reaches age 56. Therefore, the amount of money in the account is . Note that banks typically truncate, not round.

5. Based on the work you have done, write a convincing argument for what you think the value of should be.

Students might argue that since, in the context of this situation, it respresnts the fact that no 6 year periods have passed. Therefore, the account balance is the original investment of $10,000. When we use the formula and define , then which makes sense.

6. Suppose a student said that because . What would you tell this student to help them to understand the true value of Explain in detail.

Look for students to explain that an exponent indicates the number of factors of the base that we are computing. For example,

**Part 2**

Suppose you are told that someone invested some unknown amount of money at 12% annual interest so that the amount of money in the account doubled every 6 years. After 30 years, the account had a balance of $850,000.

1. How much did this person invest initially? Explain how you know.

$26,562.50

Explanations will vary but look for students to reason that to go back in time a total of 5 six year periods, they should start with $850,000 and divide by 2 a total of 5 times.

2. Suppose *n* represents the number of 6-year periods that have passed. For example, *n* = 2 means 12 years have passed or *n* = 5 means 30 years have passed. Assume that *n* = 0 corresponds to the current situation where the account balance is $850,000. Explain what each of the following expressions mean in the context of this investment situation.

a)

Since *n* = 0, this represents the current account balance of $850,000.

b)

Since *n* = 1, this represents the account balance 6 years later.

c)

Since *n* = –1, this represents the account balance 6 years prior to the current account balance.

d)

Since *n* = –2, this represents the account balance 12 years prior to the current account balance.

e)

Since *n* = –3, this represents the account balance 18 years prior to the current account balance.

3. Based on the work you have done, write a convincing argument for what you think the value of should be. What about ? What about ?

Assuming students understand that was used to find the account balance when *n* = –1 or 6 years prior to the current situation, they may realize that to determine this value they must divide by 2. Therefore, . Likewise, was used to find the account balance when *n* = –2 or 12 years prior to the current situation, then they would need to divide by 2 twice or multiply by . Therefore, . Similarly, students might discover that .

4. Write an expression that could be used to compute the amount of money used in the initial investment 30 years ago.

Since 30 years represents five 6 year periods, we can use the expression .

**Part 3**

In this part, there is no context related to money and investments. Rather, you will focus your attention on the mathematical meaning of negative exponents.

1. Complete the table.

|  |  |  |
| --- | --- | --- |
|  |  | 625 |
|  |  | 125 |
|  |  | 25 |
|  | 5 | 5 |
|  | 1 | 1 |

2. Describe the pattern you observe and write a convincing argument for why this pattern makes sense.

Look for students to describe that each subsequent row contains one less factor of 5 than the row above it. Therefore, each subsequent row can be computed by dividing the final answer in the row above it by 5. Removing a factor of 5 is equivalent to dividing by 5.

3. Complete the table.

|  |  |  |
| --- | --- | --- |
|  |  | 625 |
|  |  | 125 |
|  |  | 25 |
|  | 5 | 5 |
|  | 1 | 1 |
|  |  |  |
|  |  |  |
|  |  |  |

4. Describe the pattern you observe with the negative exponents and write a convincing argument for why this pattern makes sense.

The idea is for student to realize that, for example, is the reciprocal of .

5. Complete the table. Note that .

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  | 1 |
|  |  |
|  |  |
|  |  |

6. Describe the pattern you observe and write a convincing argument for why this pattern makes sense.

The idea is for the student to realize that, for example, is the reciprocal of .

7. Without using a calculator, compute and explain how you know.

8. Without using a calculator, compute and explain how you know.

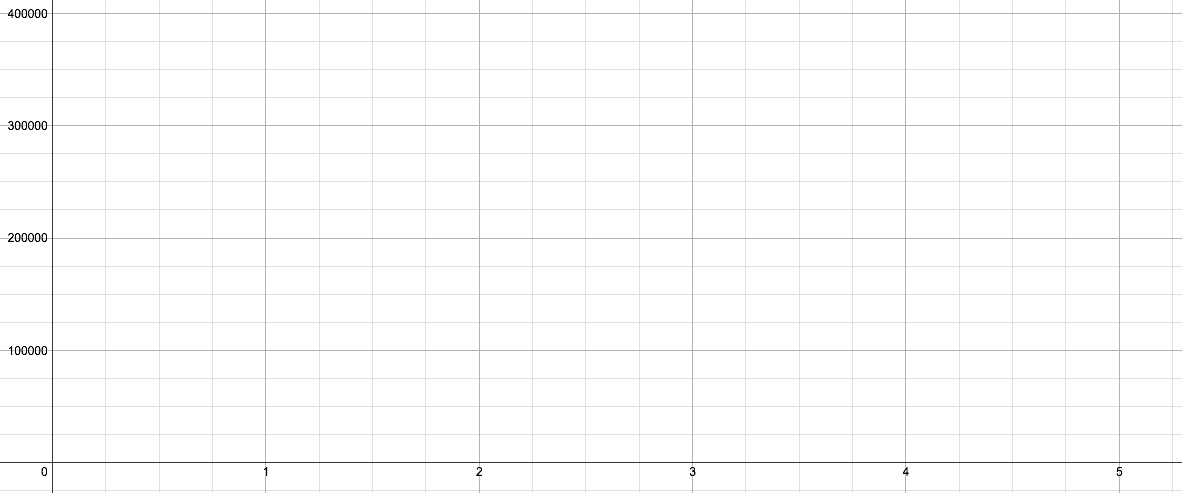
9. Explore the idea of . Write a convincing argument for how you think we should think about .

Look for reasoning related to the work done in this activity. For example, if a table of values was created like the one in Part 3, item #3, students might realize that removing a factor of 0 is equivalent to dividing by 0 which is undefined. In fact, mathematicians consider to be indeterminate. If you want more details, go to the Dr. Math Forum at <http://mathforum.org/library/drmath/view/55722.html>.

**Part 4 –** Radical and Fractional Exponents

Using the context of saving for retirement, you have had the opportunity to make sense of positive and negative exponents and zero as an exponent. What about fractional exponents like or ?

Let’s revisit the situation from Part 1 where a 35-year-old person was able to save $10,000. They plan to invest this money into a retirement savings account that earns 12% annual interest. This would mean that their money would double in value every 6 years. Suppose *n* represents the number of 6-year periods that have passed. For example, *n* = 2 means 12 years have passed or *n* = 5 means 30 years have passed.

1. In Part 1, you determined a formula for the amount of money, *A*, in the account after *n* 6-year periods is . Create a fairly accurate graph of this function.

2. Explain what each of the following expressions mean in the context of this investment situation and estimate the value of each expression using your graph.

a)

Since *n* = 0, this represents the initial investment of $10,000.

b)

Since *n* = 1, this represents the account balance after 6 years which is $20,000

c)

Since *n* = , this represents the account balance after 3 years which appears to be approximately $14,000

d)

Since *n* = , this represents the account balance after 2 years which appears to be approximately $12,500

3. Look back at 2c). Someone might claim that the amount of money in the account when *n* = is $15,000 since they expect to see an amount that is half-way between the initial amount ($10,000) and the amount when *n* = 1 ($20,000). Based on the behavior of the graph, explain why this claim is not possible.

Since the graph is not linear, we will not see an account balance of $15,000 when *n* = . Push students to explain what it means for a relationship between two quantities to be linear. In this case, we see that the amount of money in the account is not increasing at a constant rate but at an ever increasing rate. The increase in the account during the first 3-year period will be less than the increase in the account during the second 3-year period. Only if the relationship between the two quantities was linear would the increase in the first 3-year period be the same as for the second 3-year period.

4. Examine the table below.

|  |  |  |
| --- | --- | --- |
| Years since investment | *n* | Amount of money in the investment |
| 0 | 0 | $10,000 |
| 3 | 1/2 |  |
| 6 | 1 | $20,000 |

We can correctly say that $10,000 must be multiplied by some factor in order to find the amount of money after 3 years (*n* = ). Then this quantity (whatever it is) must be multiplied by the same factor in order to find the amount of money after 6 years (*n* = 1). That is, when $10,000 is multiplied by some factor twice, the result is $20,000. Write out this explanation using mathematical symbols/equations. Solve the equation and explain what the solution means in the context of this situation.

We have to multiply $10,000 by twice in order to see the investment grow to $20,000 after 6 years (*n* = 1). Therefore, the amount of money in the account after 3 years is .

5. We can write question #4 in mathematical symbols by saying to find out the amount of money in the account after 3 years (*n* = ). Based on your work in #4, what must be the value of ? Explain how you know

We want students to realize that . Push students to explain based on the idea developed in #4.

6. Using the work you did in #4 and #5, create an argument for the amount of money in the account after 2 years (*n* = )? That is, what is ?

We want students to argue that $10,000 would have to be multiplied by a factor 3 times (there are 3 copies of 2 years in 6 years) and get $20,000 after 6 years.

We can say that the account balance after 2 years is . We now can claim that .

7. In general, what is an equivalent way to express where *c* is an integer such that ?

Help students to generalize and say that .

8. Develop an argument for the following claim. That is, try to create a convincing argument to support the following statement.

For any and for an integer *c* such that , .