Exponents & Number Sense:
An Introduction to Powers of Ten

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Common Core Mathematics Standards Addressed:

- CCSS.Math.Content.5.NBT.A.2
  Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
- CCSS.Math.Content.6.EE.A.1
  Write and evaluate numerical expressions involving whole-number exponents.
- CCSS.Math.Content.8.EE.A.3
  Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other.
- CCSS.Math.Content.8.EE.A.4
  Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities. Interpret scientific notation that has been generated by technology.

Common Core Mathematics Standards Optional:

- CCSS.Math.Content.HSF.BF.B.5
  Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Materials/Setup:
∑ Review ∑ Post-it notes ∑ Number lines for millions, billions, and powers of ten ∑ PowerPoint presentation

BACKGROUND INFORMATION/REVIEW MATERIAL:
**Note that this section could take anywhere from 5 to 30 minutes depending on your students' needs. If your students need a lot of review, consider doing this review the previous day.*

In order to use power of 10 (logarithms) effectively, students will need to be effective and efficient with the rules of exponents and exponential/scientific notation. If your students need to review the rules of exponents—and they might not, depending on how recently they've studied those concepts—you might choose to have your students view this video: https://www.youtube.com/watch?v=GreZAh-18YY. Note that this video is very thorough, and as a result, is almost 14 minutes long. A suggestion would be to have your students watch the video on their own their own time so that they can view different parts of the video

[1] Cox, T. and Chen, G. (2014). [Log activity].

based on their individual needs. In addition, you might choose to let students know which parts of the video would be most relevant to them.

Here, we briefly deal with scientific notation, because it will directly apply to this introductory lesson on powers of 10 (logarithms). However, you might not need to use all or any of this review depending on the needs of your students.

The numbers used in chemistry are often either extremely large or extremely small. Such numbers can be conveniently expressed in the form $N \, x \, 10^n$ where N is a number between 1 and 10 and n is the exponent. Some examples of this ***scientific notation*** follow.

- 2,100,000 is 2.1 x 10^6 (read "two point one times ten to the sixth power")
- 0.000705 is 7.05 x 10^{-4} (read "seven point zero five times ten to the negative fourth power")

A positive exponent, as in the first example, tells us how many times a number must be multiplied by 10 to give the long form of the number:

$$2.1 \text{ x } 10^6 = 2.1 \text{ x } 10 \text{ x } 10 \text{ x } 10 \text{ x } 10 \text{ x } 10 \text{ x } 10 = 1{,}200{,}000$$

It is also convenient to think of the positive exponent as how many places we must move the decimal to the left to obtain a number greater than 1 and less than 10: If we begin with 4,350 and move the decimal point three places to the left, we end up with the scientific notation of 4.35 x 10^3.

Similarly, a negative exponent tells us how many times we must divide a number by 10 to give the long form of the number.

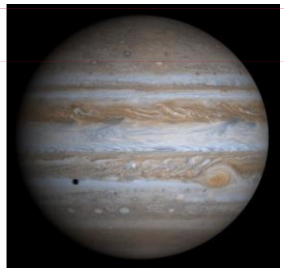
$$8.03 \text{ x } 10^{-5} = \frac{8.03}{10x10x10x10x10} = 0.0000803$$

It is convenient to think of the negative exponent as the number of places the decimal point must be moved to the right to obtain a number greater than 1 and less than 10: If we begin with 0.0048 and move the decimal point three places to the right, we end up with the scientific notation of 4.8 x 10^{-3}.

Many scientific calculators have a key labeled EXP or EE, which is used to enter numbers in scientific notation. To enter the number 5.8 x 10^3 in a calculator, the key sequence is "5.8 EXP (or EE) 3". Note that on many calculators, the display will show: 5.8 E03, or 5.8 EE3 and not the full scientific notation.

<u>THINK ABOUT IT (2 MINUTES):</u>

Space is big. REALLY big. It's so big it's almost impossible to imagine (for an amazing look at the size and scope of the universe, and to pique students' interest, you might choose to show Cosmic Voyage – Powers of Ten). What does it mean to say that the sun is 865,000 miles across? Or that Jupiter is a half-billion miles from the sun? Can you visualize that? No? No one can! The distances in space are enormous. The estimated distance to the nearest star after our own sun, Proxima (or "near") Centuari, is 4.3 light-years, which is about 25,277,549,200,000 or $2.52775492 \times 10^{13}$ miles!!!! We can, however, use mathematics as a tool to help us try to put into perspective some of these huge numbers a little bit. Let's take a look!



Jupiter
http://en.wikipedia.org/wiki/File:Jupiter_by_Cassini-Huygens.jpg

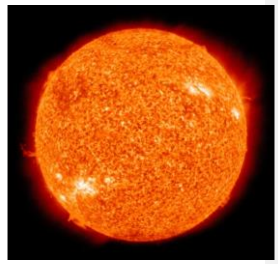
Commented [CE1]: Note that there is a PowerPoint presentation for this lesson.

Commented [CE2]: The United States' national debt is BIG too. It is measured in trillions. Check out this website for an eye-opening visual that you can share with your students: http://www.pagetutor.com/trillion/index.html

<u>THE PROBLEM WITH LARGE NUMBERS (8 MINUTES)</u>:

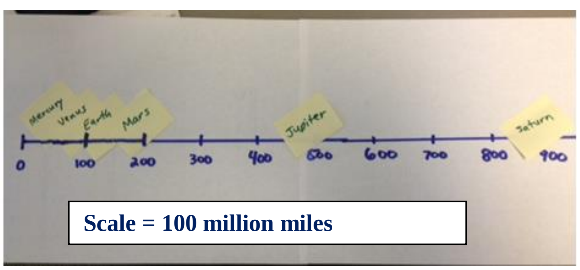
The following table[1] displays the distance from the sun to various astronomical objects (average distance, because the planets' distance varies with time depending on their orbits). The planet Mercury is 35,983,610 miles from the sun, Earth is 92,957,100 miles from the sun, and Neptune is a whopping 2,798,842,000 miles from the sun. The table also gives the distance to Proxima Centauri and Andromeda Galaxy, the spiral galaxy closest to our own galaxy, the Milky Way.

Object	Distance (million miles)	Object	Distance (million miles)
Mercury	36	Saturn	888
Venus	67	Uranus	1,784
Earth	93	Neptune	2,799
Mars	141	Proxima Centauri	25,277,549
Jupiter	484	Andromeda Galaxy	14,696,249,500,000



Sun
https://en.wikipedia.org/wiki/File:The_Sun_by_the_Atmospheric_Imaging_Assembly_of_NASA%27s_Solar_Dynamics_Observatory_-_20100819.jpg

All of the data in this table has been written on Post-it notes. When there is nobody at the board, silently get up and place your object where it belongs on the number line using units of 100 million miles provided there (see diagram below).



Andromeda Galaxy
https://en.wikipedia.org/wiki/File:Andromeda_Galaxy_(with_h-alpha).jpg

<u>DISCUSSION QUESTIONS (15 MINUTES)</u>:

Have students discuss these questions in pairs for 3 to 5 minutes.

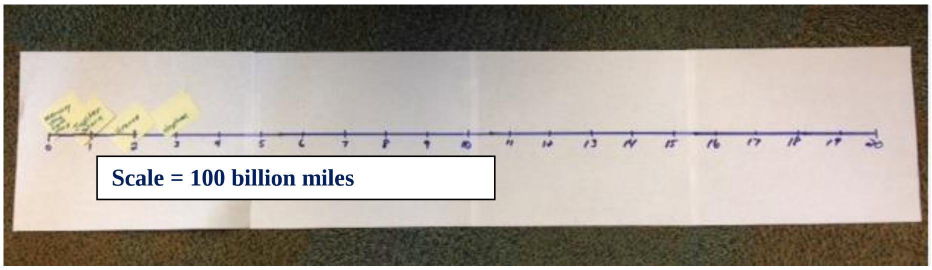
- How did you know where to place your Post-it note?
- Did you or any of your classmates have a challenge in trying to accomplish the task? Explain.
- What is the problem that we are facing? What ideas do you have about whether we can fit all of this data onto a number line?

As we have seen, we can represent the information graphically in order to get a better feel for the distances involved. However, some of the distances are very large, and we cannot even fit them onto the number line! The scale we are using is known as a linear scale, which means that the evenly spaced units represent equal distances. In each case, each unit represented 100 million miles.

The drawback is obviously that the scale is too small to show all of the astronomical distances in the table. For example, to show the distance to Proxima Centauri, we would need to go out past Neptune on the scale 8,426 times as far, and to display the Andromeda Galaxy we'd need to go out 4,898,749,833 times as far!

Commented [CE3]: Consider having students figure this out!

It would seem that we could fix this problem by choosing a larger scale. That's a good idea! Let's try that. In your groups, construct a number line with the units being 100 billion miles each and plot the distances using the Post-it notes. In a few minutes, we will have students present their findings to the class.



After presentations students should recognize that this is also a problem, because the planets are now all bunched up on top of each other. We don't see Proxima Centauri or the Andromeda Galaxy, and they are not even anywhere close to this line, even though we've made the scale 10 times larger. In fact, it would take a sheet of paper 463,896.76 miles long to get both of these objects on the same number line with the planets currently plotted!

Commented [CE4]: Again, consider having the students determine this!

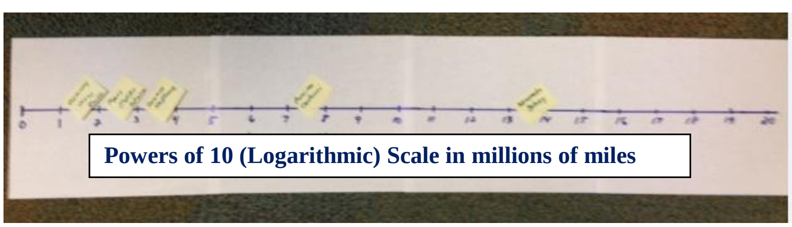
We now notice that the first five of the planets are crowded into the first unit, not allowing us to see the relative distance between them. In addition, we STILL can't fit the star and galaxy on this number line. We can conclude that a linear scale is inadequate for our needs and seek another idea!

DEALING WITH LARGE NUMBERS (10 MINUTES):

We have seen that with linear scales, if the scale is too small, more distant objects do not fit on the number line. But if it is too large, less distant objects are jammed together. We need to recognize that the problem is not that the numbers are too big or too small; the problem is that the numbers vary too greatly in size. With Mercury being "only" 36 million miles away and the Andromeda Galaxy being 14,696,249,500,000 million miles away, we have what seems to be an insurmountable challenge – enter powers of 10 (logarithms).

We consider a different type of scale: one on which equal distances are not equally spaced. This might not be ideal if our desire is to visualize relative distances, but this approach will enable us to get all of the data graphically represented.

One last time, take a few minutes and use your Post-its to plot the astronomical objects on a number line that has the scale of 10^0, 10^1, 10^2, 10^3 … million miles. Be sure to make notes of what you think "works" and what the challenges are with this approach.



Discuss what "works" and what is a "challenge". Be prepared for students to say things such as, "I can see that it's easier to get all of the objects plotted because there is more room for them to fit." or "I don't understand how making the units powers of 10 can make this much difference."

All of the objects were able to be represented for the first time. The planets are still pretty cramped, but it is at least possible to tell them apart. With this scale, we need to keep in mind that each tick mark on the scale represents a distance 10 times larger than the one before it. Consider referring back to the Cosmic Voyage – Powers of Ten video and the zooming out featured there. Notice that even though the distances are not evenly spaced, the exponents are evenly spaced. Therefore, the distances are spaced according to their powers of 10 (logarithms).

Let's look more closely at this concept.

Let's take Mars and its distance of 141 million miles. We use the fact that 100 < 141 < 1,000, so Mars's distance is between 10^2 and 10^3. To plot Mars's distance more precisely, we can use a calculator and trial and error to estimate what decimal power of 10 would give us the closest estimate for Mars' distance. We see that by restricting our answer to two decimal places our answer is approximately 2.15, and use 2.15 to represent Mars's position on the power of 10 scale.

Using this thinking, how would you explain where Saturn would be placed? How about the Andromeda Galaxy? Take a moment with your partner to show how you would answer.

CLOSING (5 MINUTES):
On a half-sheet of paper, choose one of the following synthesis activities (you can pick one for the whole class to do, or you can give students the options). Turn this in at the end of class:

- Create a Venn diagram comparing and contrasting linear and powers of 10 (logarithmic) scales/number lines.
- Write the 3 most important things you want to remember from this lesson, 2 questions you could ask yourself to check your understanding tomorrow, and 1 question you're still unsure about.

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OPTIONAL SECTION: From here on the topic of Logs are dealt with.
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Commented [CE5]: This section is ideal for advanced students or as an introduction to logarithms.

AN INITIAL LOOK AT LOGARITHMS (8 MINUTES):
Logs are like the opposite of exponents. Just like multiplication and division are opposites—they undo each other—and addition and subtraction are opposites, logs and exponents are opposites. If I were to write $a^y=x$, and I want to solve for y, $y = \log_a x$. Which means that I'm going to take the a'th log of a^y and be left with x. This is confusing for now, so let's look at an example.

$10^3 = 1{,}000$, right? If we were to do this, what's our a? 10. Our y? 3. Our x? 1000. So if I were to rearrange this equation, I'd get $3 = \log_{10}1000$. Log_{10}, which we just did with the planets, is so common that we call it our **common log**, and that's what's on your calculator. So hit log(1000), and you'll get 3. If I had $x^3=1000$, I could rearrange that as the $\log_x 1000=3$.

This is a lot, and it's new for most of us, so today we're just going to focus on rearranging and rewriting log equations so that later this week we can learn to manipulate and solve them. How could I rewrite $3^x = 9$ as a log expression? How about $\log_2 1 = x$?

The only time this isn't going to work is when we have a $\log_e$ (because remember, e is a number, too). Just like $\log_{10}$ is ridiculously common, so is $\log_e$. So much so, in fact, that we have a special name for it; we call it the natural log, ln. So we could rearrange $e^x = 8$ as $\log_e 8 = x$, but $\log_e$ becomes ln, so $\ln 8 = x$.

CLASSROOM PRACTICE (12 MINUTES):
Students complete the classwork problems on the next page. They can work with a partner or independently.
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Homework:

1. We just spent the class period dealing with large numbers and learning how powers of 10 (logarithms) can help us graphically represent them. We can also use powers of 10 (logarithms) logarithms to comprehend very small numbers. The pH of an object is a way of telling how acidic it is (note that this is a great opportunity to elicit students' knowledge of liquids that are acidic). Read and study the blog at http://blogs.scientificamerican.com/lab-rat/2012/12/03/what-makes-things-acid-the-ph-scale/. Your homework for next class is to choose 6 items off of the ruler (powers of 10 (logarithmic) scale) on that blog and explain what the pH tells us about the number of hydrogen ion concentration (**don't** worry about knowing what a hydrogen ion is…UNLESS of course, you are interested or in Chemistry!...your goal for this is to interpret the powers of 10 (logarithmic) scale).

2. Go to Secret Worlds: The Universe Within and experiment with the applet on powers of 10. Notice how each picture is actually an image of something that is 10 times bigger or smaller than the one preceding or following it. The number that appears on the lower right just below each image is the size of the object in the picture. On the lower left is the same number written in powers of ten, or ***exponential notation***. Exponential notation is a convenient way for scientists to write very large or very small numbers.

    a. Describe how many times larger the view of the Western Hemisphere of the Earth is than the view of the oak leaf.
    b. The number of times would you have to hit the "decrease" button to go from the view of our Solar System to view the cells on the leaf surface is 17. Explain using powers of ten how you would know that without actually counting the times needed.

3. Your best friend missed math class today due to the flu. Create a text message conversation between the two of you in which you summarize what we did in class today and explain the purpose of a power of 10 (logarithmic) scale and how it differs from a linear scale.