Rational Number Project

Initial Fraction Ideas*

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ABOUT THE AUTHORS

Kathleen Cramer is an associate professor in the College of Education and Human Development at the University of Minnesota. Her expertise is in the teaching and learning of mathematics in the elementary grades. She has taught mathematics in elementary school and in junior high school. Kathleen has a Ph.D. from the University of Minnesota in mathematics education. She has published articles and book chapters dealing with the teaching and learning of fractions and proportional reasoning. She has done numerous workshops for teachers dealing with fraction instruction.

Professor Cramer has been involved with the Rational Number Project (RNP) since 1980. She participated in the initial teaching experiments with fourth and fifth graders and has been co-PI on the NSF grants since 1984. She has taken the primary responsibility for revising the lessons developed from the research to form the two sets of RNP Fraction Lessons for the Middle Grades.

Merlyn Behr was for over 25 years a professor of mathematics education at Northern Illinois University in DeKalb, Illinois. He was also a faculty member at Florida State University where he received his Ph.D., and at Louisiana State University at Baton Rouge. Merlyn's primary interest was in children's learning of elementary- and middle-grades mathematical concepts. He contributed a great deal to our understanding of children's cognitive processes in these areas. He was very active in the research community and served on the editorial board of the Journal for Research in Mathematics Education (JRME) and as chair of the North American chapter of the research group of the Psychology of Mathematics Education. As a co-founder of the RNP, Merlyn was instrumental in charting its course and providing much valued intellectual leadership in many aspects of RNP activity.

Merlyn died in February 1995. His wit and professional contributions are sorely missed.

Thomas Post, former high school mathematics teacher in New York State, joined the faculty of the College of Education at the University of Minnesota in 1967 after receiving his Ed.D. from Indiana University. Professor Post's interest is closely allied with other RNP members, as he is especially interested in children's and teachers' perceptions of middle-school mathematics. He also has an interest in interdisciplinary approaches to curriculum. He was a co-founder of the RNP and has been active in the mathematics education research community. Along with Kathleen Cramer, Merlyn Behr and Richard Lesh, he has been one of the co-authors of some 70 papers, book chapters and technical reports produced by the RNP since it's inception in 1979. Tom has also served on the editorial board of the JRME and has been chair of the North American chapter of the research group Psychology of Mathematics Education.

Richard Lesh, former professor and dean at Northwestern University, received his Ph.D. from Indiana University. He spent 5 years overseeing computer software development in mathematics and science at WICAT systems in Provo, Utah. He then served as senior research scientist at ETS in Princeton, NJ where he developed innovative strategies and materials for assessing outcomes in mathematics classrooms. Professor Lesh has served as project manager of the program unit - Research on Teaching and Learning - at the National Science foundation. Currently, he is a Professor of Learning Science and Director of Center for Research on Learning and Technology at Indiana University, and is a leading researcher in the area of implementing models and modeling in STEM curriculum – particularly in mathematics, research methodologies, and complex assessment systems. Dr. Lesh is one of the original co-founders of the RNP and has worked on its first six grants since 1979.

Merlyn Behr was a founding member of the Rational Number Project. He developed the first draft of the fraction materials that was used in the initial research with children. His insight into children's learning, and his thoughtful analyses if students' thinking are an integral part of this curriculum.

These Lessons are dedicated to Merlyn Behr. After a long and illustrious career in mathematics education, he passed away in February, 1995.

The authors would like to acknowledge the important contributions made by the fourth- and fifth-grade teachers in the Rosemount-Apple Valley, MN school district who participated in the pilot study of these materials. Their suggestions for improving the lesson have been incorporated into the materials. The quality of these lessons is due in part from the input these teachers provided.

Content

III. Appendix – Fraction Circles and Assessment Ideas

- Pictures of the Fractions
- Fraction Circle Masters
- Parent and Child Activity
- Quizzes
- Written Test
- Student Interview 1
- Student Interview 2
- Student Interview 3
- Student Interview 4

Preface

The **Rational Number Project** (RNP) is a cooperative research and development project funded by the National Science Foundation. Project personnel have been investigating children's learning of fractions, ratios, decimals and proportionality since 1979. This **book of fraction lessons** is the product of several years of working with children in classrooms as we tried to understand how to organize instruction so students develop a deep, conceptual understanding of fractions.

The lessons were originally published in 1997 under the title: RNP: *Fraction Lessons for the Middle Grades Level 1*. They have been revised (August, 2009) and renamed to better reflect its content. *Initial fraction ideas* include developing meaning for fractions using a part-whole model, constructing informal ordering strategies based on mental representations for fractions, creating meaning for equivalence concretely, and adding and subtracting fractions using concrete models. Initial fraction ideas do not include formal algorithms, and instruction with formal algorithms was not part of this original RNP curriculum module.

A companion module has been developed with NSF support. This module, *Fraction Operations and Initial Decimal Ideas*, extends students' fraction ideas to develop fraction operations of addition, subtraction, multiplication and division with symbols. The new module also introduces students to decimal ideas – naming decimals, order, equivalence, addition and subtraction. This module can be found on the RNP website at this address:

http://www.cehd.umn.edu/rationalnumberproject/rnp2.html

- **These lessons** provide teachers with an alternative to the textbook scope and sequence for fraction instruction and are appropriate for students in grades 4 – 6 but have also been shown to be effective in remedial settings with older students.

- **These lessons** help students develop number sense for fractions because they invest time in the development of concepts, order and equivalence ideas.

- **These lessons** provide students with daily "hands-on" experiences. Fraction circles, chips and paper folding are the manipulative models used in these lessons to develop initial fraction ideas.

- **These lessons** provide teachers with daily activities that involve children in large group and small group settings. All the lessons involve students using manipulative materials. Our work with children has shown that students need extended periods of time with manipulatives to develop meaning for these numbers.

- **These lessons** offer teachers insight into student thinking as captured from the RNP research with children. The "Notes to the Teacher" section shares examples of students' misunderstandings, provides anecdotes of student thinking, and contains information on using manipulative materials.

- **These lessons** will help teachers and students attain the goals set for fractions by the National Council of Teachers of Mathematics in their *Principals and Standards for School Mathematics (2000).*

- Understand fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
- Use models, benchmarks, and equivalent forms to judge the size of fractions;
- Recognize and generate equivalent forms of commonly used fractions;
- Develop and use strategies to estimate computations involving fractions;
- Use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions.

- **These lessons** delay development of operations with fractions until students have developed meaning for fractions. The lessons develop understanding of the operations with fractions by using story problems and fraction circles. Estimation is emphasized throughout.

- **These lessons** have been used with over 1600 students. Teachers' response to them has been uniformly enthusiastic. Results of pilot testing show significant differences in students' performance between students who used these lessons as compared to students who used the textbook for fraction instruction. Students using the RNP lessons outperformed students using the district's commercial textbook. Marked differences in thinking were noted in the pilot study. Students using the RNP lessons thought about fractions in a conceptual manner, while students using the textbook thought about fractions in a procedural manner.

- **These lessons are for teachers who know from their own experience with children that there must be a better way to teach fractions!!! Students who use these lessons will develop concepts and will be able to operate on fractions meaningfully.**

Teacher's Guide
RNP: Initial Fraction Ideas

Introduction

These lessons reflect research on children's fraction learning conducted by the National Science Foundation-sponsored Rational Number Project (RNP). Since 1978, Merlyn Behr (Northern Illinois University), Kathleen Cramer (University of Minnesota), Thomas Post (University of Minnesota) and Richard Lesh (Indiana University) have studied how elementary-aged children learn to extend their understanding of numbers to include fraction ideas.

The RNP staff conducted teaching experiments with fourth- and fifth-grade children. In a teaching experiment, researchers entered the classroom as teachers and worked with a curriculum created from a well thought-out theoretical framework. During the teaching and learning process, researchers study children's learning as they progress through the lessons. This is done through classroom observations, student interviews and written assessments.

The curriculum created for the teaching experiments that evolved into these lessons reflected the following beliefs: (a) Children learn best through active involvement with multiple concrete models, (b) physical aids are just one component in the acquisition of concepts: verbal, pictorial, symbolic and real-world representations also are important, (c) children should have opportunities to talk together and with their teacher about mathematical ideas, and (d) curriculum must focus on the development of conceptual knowledge prior to formal work with symbols and algorithms.

The teaching experiments were conducted in two parts. The first phase of the project was ten weeks long with small groups of fourth and fifth graders at two different sites (Minnesota and Illinois). The second phase was conducted with a classroom of 30 students. A class of fourth graders participated from January of their fourth grade, through January of their fifth grade. Instruction was four days per week and covered the following topics: part-whole model for fractions, ratio and quotient models for fractions, decimals, and number lines.

Throughout both teaching experiments a subset of children were interviewed every two weeks. The interviews provided information on children's thinking about fraction ideas. We were interested in what role manipulative materials played in their thinking as well as what understandings and misunderstandings children have with fractions.

Our work with children helped explain why children have so much difficulty with fractions. It also informed us as to the type of experiences children need to develop a deep, conceptual understanding of fractions. Consider a few of the insights garnered from these teaching experiments:

1. Children have difficulty internalizing that the symbol for a fraction represents a single entity. When asked if $\frac{2}{3}$ was one or two numbers, many children would say that the symbol represented two numbers. When students consider $\frac{2}{3}$ as two numbers then it makes sense to treat them like whole numbers. For example, when students add two fractions by adding the numerators and then denominators, they are interpreting the symbols as four numbers, not two. Many errors with fractions can be traced to students' lack of mental images for the quantity the symbol represents.

2. Ordering fractions is more complex than ordering whole numbers. Comparing $\frac{1}{4}$ and $\frac{1}{6}$ conflicts with children's whole number ideas. Six is greater than four, but $\frac{1}{4}$ is greater than $\frac{1}{6}$. With fractions, **more** can mean **less**. The **more** equal parts you partition a unit into, the **smaller** each part becomes. In contrast, $\frac{3}{5}$ is greater than $\frac{2}{5}$ because 3 of the same-size parts are greater than 2 of the same-size parts. In this case, **more** implies **more**. Being able to order plays an important part in estimating fraction addition and subtraction. Ideally when a student adds, for example, $\frac{1}{4}+\frac{1}{3}$, she should be able to reason from her mental images of the symbols that (a) the answer is greater than $\frac{1}{2}$, but less than one and (b) $\frac{2}{7}$ is an unreasonable answer because it is less than $\frac{1}{2}$.

3. Understanding fraction equivalence is not as simple as it may seem. Some children have difficulty noting equivalence from pictures. Imagine a circle partitioned into fourths with one of those fourths partitioned into three equal parts. Some children we worked with were unable to agree that $\frac{3}{12}$ equals $\frac{1}{4}$ even thought they agreed that physically the two sections were the same size. Children said that once the lines were drawn in, you could not remove them. [Therefore $\frac{3}{12} \neq \frac{1}{4}$]. In reality, that is just what must be done to understand fraction equivalence from a picture.

4. Difficulties children have with fraction addition and subtraction come from asking them to operate on fractions before they have a strong conceptual understanding for these new numbers. They have difficulty understanding why common denominators are needed so they revert to whole number thinking and add numerators and denominators.

The RNP Curriculum

The RNP curriculum offers an alternative scope and sequence to one suggested in fourth- or fifth-grade textbooks. The RNP philosophy is that extended periods of time invested with manipulative materials developing concepts, order, and equivalence ideas are needed before students can operate on fractions in a meaningful way. We call these skills, initial fraction ideas. These goals are consistent with the instructional goals set forth in the National Council of Teachers of Mathematics in their *Principles and Standards for School Mathematics*. The RNP curriculum provides teachers with carefully researched lessons to meet these goals.

The RNP Level 1 materials develop the following topics: (a) part-whole model for fractions, (b) concept of unit, (c) concepts of order and equivalence and (d) addition and subtraction of fractions at the concrete level. The concrete models used are fraction circles, paper folding and chips. It de-emphasizes written procedures for ordering fractions, finding fraction equivalences, and symbolic procedures for operating on fractions. Instead it emphasizes the development of a quantitative sense of fraction.

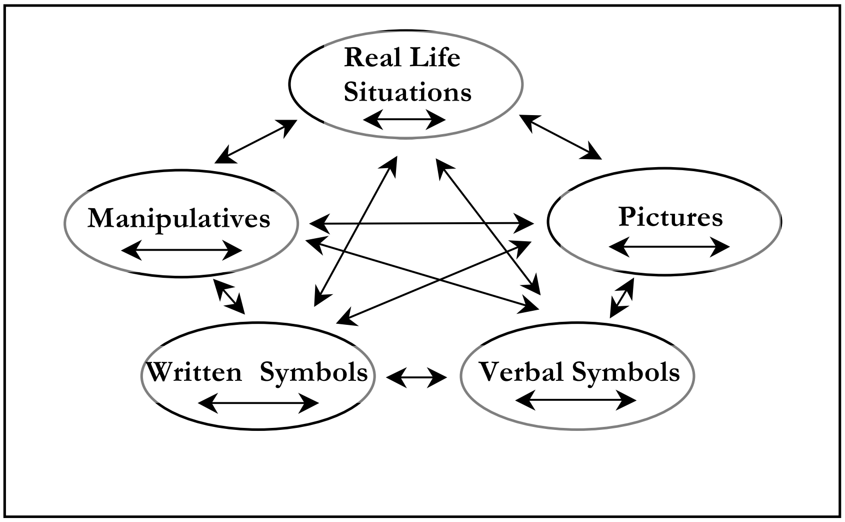
To think quantitatively about fractions, students should know something about the relative size of fractions and be able to estimate a reasonable answer when fractions are operated on. Below, find an example of a fourth-grad student's thought process for estimating a fraction addition problem. This student used the RNP curriculum; her thinking reflects a quantitative sense of fraction. Students using the RNP lessons develop this type of understanding for fractions.

> **Problem: John calculated the problem as follows: 2/3 + 1/4 = 3/7. Do you agree?**
>
> *Student*: I don't agree. He did it weird. You don't add the top numbers and bottom numbers.
>
> *Teacher*: What would be an estimate?
>
> *Student*: It would be…greater than 1/2 because 2/3 is greater than 1/2.
>
> *Teacher*: Would it be greater or less than one?
>
> *Student*: Less than one. You'd need 1/3 and 1/4 is less than 1/3.
>
> *Teacher*: What about 3/7?
>
> *Student*: 3/7 is less than 1/2.
>
> *Teacher*: How do you know?
>
> *Student*: Because 3/7 isn't 1/2. I just know.

Theoretical Framework

Children using these lessons will be using several manipulative models and will consider how these models are alike and different. They will work in small groups talking about fraction ideas as well as interacting with the teacher in large group settings. They will be drawing pictures to record their actions with fraction models. They will be solving story problems using manipulatives to model actions in the stories.

This model for teaching and learning reflects the theoretical framework suggested by Jean Piaget, Jerome Bruner, and Zoltan Dienes. Richard Lesh, a long time RNP member, suggested an instructional model that clearly shows how to organize instruction so children are actively involved in their learning. Consider this picture.



Lesh suggests that mathematical ideas can be represented in the five ways shown here. Children learn by having opportunities to explore ideas in these different ways and by making connections between different representations. This model guided the development of the RNP curriculum.

Lesson Format

The lessons reflect a classroom organization that values the important role a teacher plays in student learning as well as the need for students to work cooperatively, talking about ideas, and using manipulative models to represent rational number concepts. Each lesson includes an overview of the mathematical idea developed. Materials needed by teachers and students are noted. The lesson begins with a class **Warm Up**. Warm Ups are used to review ideas developed in previous lessons and should take only 5-10 minutes of class time. There is a **Large Group Introduction** section in each lesson. The teacher's lesson plans provide problems and questions to generate discussion and target the exploration. **Small Group/Partner Work** is included in each lesson where students together continue the exploration of ideas introduced in the large group. The class ends with a **Wrap Up**. A final activity is presented to bring closure to the lesson. At times this will be a presentation by students of select problems from the group work. We found that students like to share their thinking. At other times the Wrap Up will be another problem to solve as a group. The amount of time needed for each lesson will vary from classroom to classroom. A single lesson does not necessarily reflect one day's work, though teachers often will find that one day is sufficient to cover the material.

An important part of each lesson is the "Comments" section. Here insights into student thinking captured from the initial RNP teaching experiments are communicated to teachers. These notes clarify a wide variety of issues, such as why mastery at the symbolic level is not the primary objective for many of the earlier lessons. The notes also share examples of students' misunderstandings for teacher's reflection and anecdotes of student thinking from earlier RNP projects. These notes to the teachers also clarify methods for using manipulative materials to model fraction ideas.

Manipulative Materials

Fraction circles, two-sided colored counters and paper folding are the manipulative models used. Our research has shown that the fraction circles are the most important manipulative model for developing mental images of fraction symbols.

Fraction Circles

The master for the fraction circles are in the appendix with a page showing the different partitions and colors used for the fraction circles. The circles should be duplicated on index using colors noted on each master. Teachers who have used the fraction circles have relied on their students, parents or teacher-aids to cut out the circles and to organize them in two-pocket folders. If you choose to send home the fraction circles to be cut out with the parent's help, you will find in the appendix a parent and child activity sheet for them to do together once the circles are cut out.

Counters

Two-sided colored counters are available from most publishers of mathematics manipulative materials. A less expensive way is to purchase from a tile store, one square inch tiles (white on one side, tan on other). These cost less than 1.5 cents per tile. Thirty per student should be enough.

Paper Folding

Use 8.5″ by 11″ sheets of paper cut into strips 1″ by 8.5″. Have lots on hand for students to use for lessons 7 and 10.

Pilot Testing

The RNP Level 1 lessons have been piloted in the Rosemount-Apple Valley, MN school district. Some 66 teachers participated in this study. Thirty-three fourth- and fifth-grade teachers used the RNP lessons while thirty-three other fourth- and fifth-grade teachers used the textbook. RNP students outperformed textbook students in all areas assessed. Particular differences were noted in students' thinking. RNP students thought about fractions in a conceptual manner, while textbook students generally thought about fractions procedurally. RNP students were much more able to verbalize their thinking about fractions than textbook students.

Teachers from this pilot who used the RNP curriculum gave us detailed feedback on lessons. The lessons have been revised to reflect the input from these teachers.

Assessment

Four quizzes are provided in the appendix to assess students' fraction understanding as they work through all 23 lessons. A written test is provided for use at the end of the lessons. Items reflect information directly taught in the lessons as well as extensions. Another important source of information on children's thinking can come from talking to students. Interviews are included to be used at certain intervals in the lessons. You may want to select three students representing a range of understanding and interview them as they progress through the lessons. The interview items come from the interviews we used with children and reflect questions we feel gave us the most information on children's thinking.

Interview questions assess understanding on content addressed in the lessons as well as questions on content to be addressed in upcoming lessons. It is interesting to see how students transfer information to new situations. Of particular interest is interview #3 which, if used, should be given before lesson 19. This interview asks students to add and subtract fractions before the lessons on addition and subtraction. We found it interesting to note how many students were able to construct their own reasonable process for adding and subtracting fractions. A written test and four interviews can be found in the appendix.

Special Notes on Students' Thinking

From our interviews with children we noted that they constructed what we now refer to as informal strategies for ordering fractions. These strategies reflect students' use of mental images of fractions to judge the fraction's relative size. These informal strategies do not rely on procedures usually taught: least common denominators and cross-products. We have named the four strategies noted in students' thinking as: same numerator, same denominator, transitive and residual strategies.

When comparing $\frac{2}{3}$ and $\frac{2}{6}$ (fractions with the same numerator) students can conclude that $\frac{2}{3}$ is the larger fraction because thirds are larger than sixths and two of the larger pieces must be more than two of the smaller pieces. This strategy involves understanding that an inverse relationship exists between the number of parts a unit is partitioned into and the size of the parts.

The same denominator strategy refers to fractions like $\frac{3}{8}$ and $\frac{2}{8}$. In this case, the same denominator implies that one is comparing parts of the unit that are the same size. Three of the same-size parts are greater than two of the same-size parts.

The student strategy that has been termed the *transitive* strategy can be modeled by comparing $\frac{3}{7}$and $\frac{5}{9}$. When making this comparison, a student can conclude that $\frac{3}{7}$ is less than $\frac{5}{9}$ because $\frac{3}{7}$ is less than $\frac{1}{2}$, while $\frac{5}{9}$ is greater than $\frac{1}{2}$. This is the transitive strategy because students use a single outside value to compare both fractions.

When comparing $\frac{3}{4}$ and $\frac{5}{6}$, a student can reflect that both fractions are one "piece" away from the whole unit. Because $\frac{1}{6}$ is less than $\frac{1}{4}$, $\frac{5}{6}$ must be closer to the whole and is therefore the bigger fraction. This thinking strategy has been called a *residual* strategy because students focus on the part "leftover" in judging the relative size of the fractions.

These four strategies closely parallel students' actions with manipulatives. They are in contrast to the paper and pencil procedures, which require changing both fractions to common denominators or calculating cross-products. RNP lessons developed only these student-constructed strategies. The order questions on the interviews will assess whether students construct these strategies. Students who have constructed these strategies have developed are on the way to developing number sense for fractions.

Final Comments

You will find at the end of each lesson a form for you to record your adaptations for each lesson. Any curriculum will need to be "personalized" by the teacher who uses it, so it best meets the needs of his/her students. This form will act as a reminder about changes you feel are important to make the next time you teach the lesson.

The RNP Lessons
Initial Fraction Ideas

RATIONAL NUMBER PROJECT
Initial Fraction Ideas
Scope and Sequence

LESSON	MANIPULATIVE	TOPIC
1	Fraction Circles	Exploration with the circles.
2	Fraction Circles	Model and verbally name: 1-half, 1-third, 1-fourth.
3	Fraction Circles	Model and verbally name unit fractions with denominators greater than 4.
4	Paper Folding	Compare paper folding to fraction circles. Model and name (verbally and with written words) unit and non-unit fractions.
5	Fraction Circles	Model fractions and record with symbols a/b.
6	Fraction Circles	Model the concept that the greater the number of parts a unit is divided into, the smaller each part is.
7	Paper Folding	Reinforce the concept that the greater the number of parts a unit is divided into, the smaller each part is.
8	Fraction Circles	Fraction Equivalence
9	Fraction Circles Pictures	Fraction Equivalence
10	Paper Folding	Fraction Equivalence.
11	Fraction Circles	Order fractions by comparing to 1-half.
12	Chips	Introduce new model for fractions less than one by comparing to a familiar model.

LESSON	MANIPULATIVE	TOPIC
13	Chips	Model fractions using several units for the same fraction.
14	Chips	Model fractions using chips; determine fractions that can be shown given a set of chips.
15	Chips	Fraction Equivalence.
16	Fraction Circles	Reconstruct the unit given the fraction part.
17	Fraction Circles	Model fractions greater than one sing mixed and improper fraction notation.
18	Fraction Circles	Fraction equivalence for 1-half based on a number pattern.
19	Fraction Circles	Estimate sum of two fractions within story contexts.
20	Fraction Circles	Find the sum of two fractions using fraction circles.
21	Fraction Circles	Estimate and solve concretely fraction subtraction using "take-away" and "difference" contexts.
22	Fraction Circles	Estimate and solve fraction subtraction using "difference" and "how many more" contexts.
23		Summary activities to tie together students' number sense and addition and subtraction.