**Number Systems**
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I. **Introduction**

Have you ever noticed that certain numbers pop up over and over again in computers? Numbers like 32, 256, 512, and 1024 are commonplace, but why not just 30, 250, 500, or 1000? Why is it a 32 GB flash drive instead of a 30 GB? That's a great question! Let's take a look.

We have all learned to count using the decimal system. That means we have 10 numerals (deci meaning ten) to use, 0-9. However, computers use a different system called binary. This uses only two numerals – 0 and 1. Everything that a computer processes: words, letters, colors, pictures, is converted to those two numbers through switches on a computer chip. In actuality, to a computer the number "0" means "off" and the number "1" means "on". The computer either recognizes a signal or not. This is the reason computers were designed to work with the binary system. We don't have any way to have a switch "halfway on."
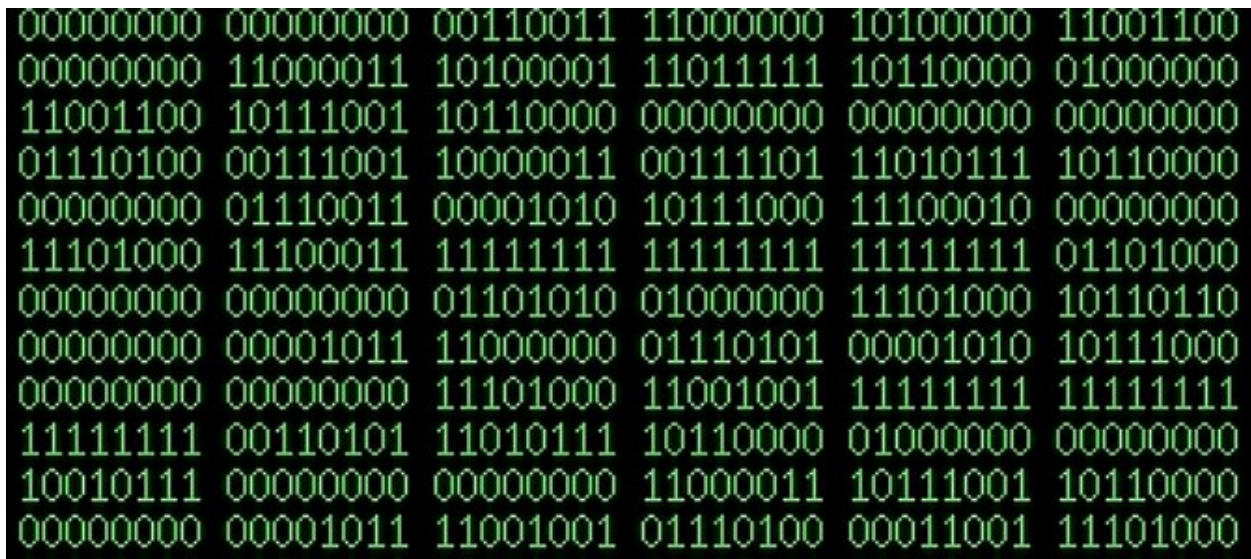
II. **Investigating Binary Numbers – Binary to Decimal**

An example of a binary number is 10011_2 (Note: We can use a subscript to indicate the base we are using but if it is clear from the context what base we are in then the subscripts are omitted.)

To figure out the decimal (base 10) equivalent of 10011_2, we should review place value.

Similar to the decimal system, the place value of binary numbers get larger as you move to the left. But instead of multiplying by the base of ten for each successive place value, you multiply by the base of two. Think about how the decimal system is set up.

First, there is the one's place, then 1 x 10 or the ten's place, then the 1 x 10 x 10 or the hundred's place.

10,000 (1x10x10x10×10)	1,000 (1x2x2×2)	100 (1x2×2)	10 (1×10)	1 (1)	.
					decimal

Binary follows the same pattern, except you have one's place, then 1X2 or the two's place, then the 1 x 2 x 2 or the four's place, and so on:

16 (1x2x2x2×2)	8 (1x2×2x2)	4 (1x2×2)	2 (1×2)	1	.
1	0	0	1	1	"binimal?" ☺

If there is a one in a specific place value we count 1 of that number and we add all of the numbers together to get the base 10 value. In this example of 10011_2, we get…
$16 + 0 + 0 + 2 + 1 = 19_{10}$.

You can extend the place values out as far as necessary, each time multiplying the previous number by 2. After 16, you'd have the following place values in base 2: 32, 64, 128, 256, 512, 1024. You may recognize that these values show up over and over again in computing, especially in the past when we had smaller values of products like memory cards and flash drives.
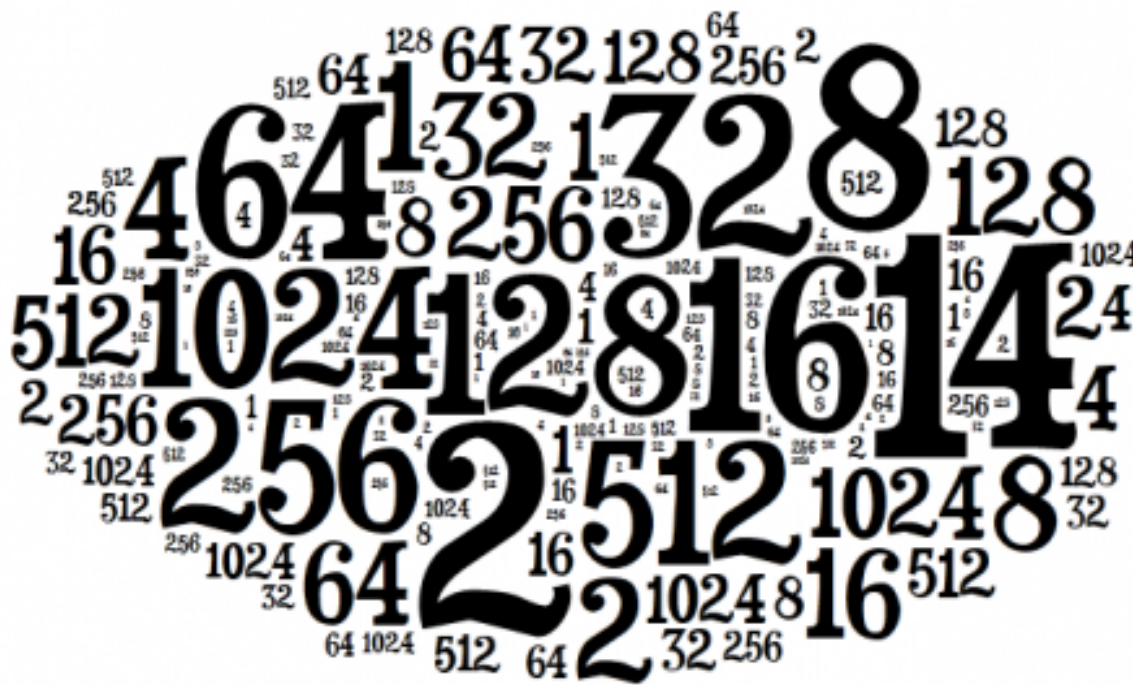
III. Investigating Binary Numbers – Decimal to Binary

You can also take a decimal number and convert it to binary. For example let's start with 55. You would need to consider the binary place values (1, 2, 4, 8, 16, 32, 64, etc.) and find the largest one that would go into the number. In this case, 32. Place a 1 in the 32nds place value.

32 (1x2x2x2x2×2)	16 (1x2x2x2×2)	8 (1x2×2x2)	4 (1x2×2)	2 (1×2)	1	.
1						"binimal?" ☺

Then find the difference between the numbers to see what quantity we have remaining. We see the value left is 23. Then we repeat the process, placing a "1" in the 16s place.

32 (1x2x2x2x2×2)	16 (1x2x2x2×2)	8 (1x2×2x2)	4 (1x2×2)	2 (1×2)	1	.
1	1					"binimal?" ☺

We now have 7 remaining: 23 – 16 = 7. We continue this process until the remaining value is 0. We get 7 – 4 = 3, 3 – 2 = 1, and 1 – 1 = 0.

32 (1x2x2x2x2×2)	16 (1x2x2x2×2)	8 (1x2×2x2)	4 (1x2×2)	2 (1×2)	1	.
1	1	0	1	1	1	"binimal?" ☺

IV. Problems to Contemplate

1. Using the process of determining increasingly smaller place values to the right of the decimal point in the decimal system, what would the first four place values be in base 2? Base 8? Explain.

2. What would the value 2304.00321_5 be equal to in base 10?

3. What would the number 65402_7 be equal to in base 4?

4. $324_6 - 115_6 =$

5. $530_7 + 612_7 =$

6. $8_9 \times 3_9 =$

7. $\dfrac{12_3}{2_3}$

V. History of Numbers

A. Tally System

Counting is a universal human activity. However, most people do not begin their counting with the zero. Counting first began when people would use their fingers to keep track of "how many", even a small child can hold up fingers and say "I am this many". What would happen when you turn 11? No more fingers, perhaps you could include the toes but we don't have an unlimited number amount of toes either. That is why many groups of people invented their own "Tally System", which is still in common use today. In this system, you make a small vertical line for one unit, then another next to it for the second, and so on until you reach the fifth – which crosses the other four. You then start a over on the next set of five.

One of the simplest ways to track quantities other than using one's fingers (and toes) is to use tallies. For instance, if you are trying to count how many times your teacher says the phrase "um" when he/she is teaching you could use tallies. For the first "um" you would mark:

I

For the 2nd "um:

II

The 3rd and 4th:

IIII

And the 5th we typically cross of the set of 5:

卌

Of course this process could be continued for as long as the teacher uses that now overused and annoying phrase! How many "um's" would be uttered with this many tallies?

卌
卌
卌
卌
卌
卌
III

This turns out to be a great system for keeping track of a small number of objects over time when you may need to add another at any time. Of course it quickly becomes unwieldy to repeatedly count several thousand tallies.

B. Roman Number System (http://www.archimedes-lab.org/numeral.html)

Before adopting the Hindu-Arabic numeral system we use today, people used the **Roman figures** instead. The Roman numeration is based on a biquinary (5) system.

To write numbers the Romans used an additive system: **V** + **I** + **I** = **VII** (7) or **C** + **X** + **X** + **I** (121), and also a subtractive system: **IX** (**I** before **X** = 9), **XCIV** (**X** before **C** = 90 and **I** before **V** = 4, 90 + 4 = 94). Latin numerals were used until the late XVI century!

The graphical origin of the Roman numbers
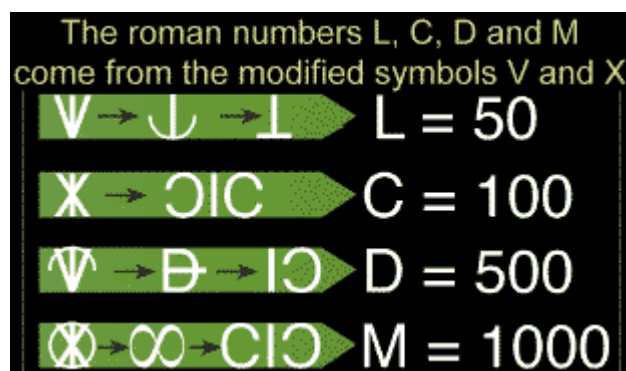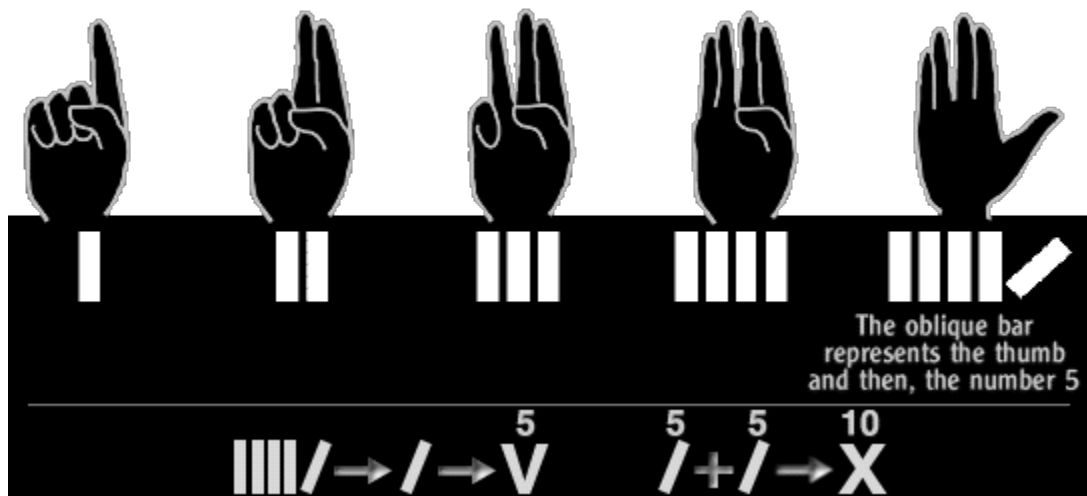
It is remarkable that the rise of a civilization as advanced as Alexandria also meant the end of a place-value number system in Europe for nearly 2,000 years. Neither Egypt nor Greece nor Rome had a place-value number system, and throughout medieval times Europe used the absolute value number system of Rome (Roman Numerals). This held back the development of mathematics in Europe and meant that before the period of Enlightenment of the 17th century, the great mathematical discoveries were made elsewhere in East Asia and in Central America.

C. The Rise of Zero, Place Value, and the Decimal Number System
Zero is often equated with "nothing," but that is not a good analogy. Zero can be the absence of a quality, but it can also be a starting point. On the number line zero is exactly between the negative and positive numbers. The zero is the starting point or origin on a graph with an x-axis and a y-axis. The word origin sounds very important. The origin is the beginning. Well, so is the zero. It was so important mathematicians

actually feared it and refused to use it. It came in and out of history and it is really hard to pin-point the exact moment the zero was created.

The origin of the modern decimal-based place value system is ascribed to the Indian mathematician Aryabhata I, 498 AD. Using Sanskrit (the primary language of Hinduism) numeral words for the digits, Aryabhata stated "Sthanam sthanam dasa gunam" or "place to place is ten times in value."

The earliest recorded inscription of decimal digits to include the symbol for the digit zero, a small circle, was found at the Chaturbhuja Temple at Gwalior, India, dated 876 AD. This Sanskrit inscription states that a garden was planted to produce flowers for temple worship and calculations were needed to assure they had enough flowers. Fifty garlands are mentioned and in this document both the values 50 and 270 are written with zero. It is accepted as the undisputed proof of the first use of zero. (http://www.vedicsciences.net/articles/history-of-numbers.html)

Today's numbers, also called Hindu-Arabic numbers, are a combination of just 10 symbols or digits: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. These ten digits were introduced in Europe within the XII century by Leonardo Pisano (Fibonacci), an Italian mathematician. Fibonacci was educated in North Africa, where he learned and later carried to Italy the now popular Hindu-Arabic numerals. Hindu numeral system is a pure place-value system which is one reason why zero is needed – as a place holder.

The Indian numerals are elements of Sanskrit and existed in several variants well before their formal publication during the late Gupta Period (320-540 AD). In contrast to all earlier number systems, the Indian numerals did not relate to fingers, pebbles, sticks or other physical objects.

Brahmi			—	=	≡	+	௶	℮	ꓶ	Ƨ	?
Hindu		०	१	२	३	४	५	६	७	८	९
Arabic		٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Medieval		O	I	2	3	୧	୨	6	ୱ	8	9
Modern		0	1	2	3	4	5	6	7	8	9

The development of this system hinged on three key abstract (and certainly non-intuitive) principles:

a. The idea of attaching to each basic figure graphical signs which were removed from all intuitive associations, and did not visually evoke the units they represented;
b. The idea of adopting the principle according to which the basic figures have a value which depends on the position they occupy in the representation of a number; and
c. The idea of a fully operational zero, filling the empty spaces of missing units and at the same time having the meaning of a null number.

The Indian number system is exclusively a base 10 system, in contrast to the Babylonian (modern-day Iraq) system, which was base 60; for example, the calculation of time in seconds, minutes and hours. By the middle of the 2nd millennium BC, Babylonian mathematics had a sophisticated sexagesimal positional numeral system (based on 60, not 10).

D. Mayan Number System (http://www.math.wichita.edu/history/topics/num-sys.html)

The Mayan number system dates back to the fourth century and was approximately 1,000 years more advanced than the Europeans of that time. This system is unique to our current decimal system, which has a base 10, in that the Mayan's used a vigesimal system, which had a base 20. This system is believed to have been used because, since the Mayan's lived in such a warm climate and there was rarely a need to wear shoes, 20 was the total number of fingers and toes, thus making the system workable. Therefore two important markers in this system are 20, which relates to the fingers and toes, and five, which relates to the number of digits on one hand or foot.

The Mayan system used a combination of two symbols. A dot (.) was used to represent the units (one through four) and a dash (-) was used to represent five. It is thought that the Mayan's may have used an abacus because of the use of their symbols and, therefore, there may be a connection between the Japanese and certain American tribes (Ortenzi, 1964). The Mayan's wrote their numbers vertically as opposed to horizontally with the lowest denomination on the bottom. Their system was set up so that the first five place values were based on the multiples of 20. They were 1 (20^0), 20 (20^1), 400 (20^2), 8,000 (20^3), and 160,000 (20^4). In the Arabic form we use the place values of 1, 10, 100, 1,000, and 10,000. For example, the number 241,083 would be figured out and written as follows:

Mayan Numbers	Place Value	Decimal Value
•	1 times 160,000	= 160,000
≡	10 times 8,000	= 80,000
• •	2 times 400	= 800
•••• ≡	14 times 20	= 280
• • •	3 times 1	= 3

The Mayan's were also the first to symbolize the concept of nothing (or zero). The most common symbol was that of a shell (see below) but there were several other symbols (e.g. a head). It is interesting to learn that with all of the great mathematicians and scientists that were around in ancient Greece and Rome, it was the Mayan Indians who independently came up with this symbol, which usually meant completion as opposed to zero or nothing. Below is a visual of different numbers and how they would have been written:
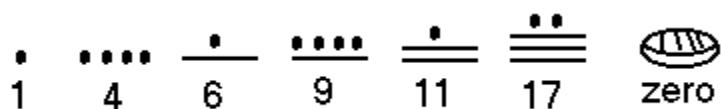
In the table below are represented some Mayan numbers. The left column gives the decimal equivalent for each position of the Mayan number. Remember the numbers are read from bottom to top. Below each Mayan number is its decimal equivalent.

8,000						•••
400			•	•	••	• ≡
20	•	••	••	——	••	shell
units	shell	shell	——	•••	••• ≡	•••• ≡
	20	40	445	508	953	30,414

It has been suggested that counters may have been used, such as grain or pebbles, to represent the units and a short stick or bean pod to represent the fives. Through this system the bars and dots could be easily added together as opposed to such number systems as the Romans but, unfortunately, nothing of this form of notation has remained except the number system that relates to the Mayan calendar.

E. Egyptian Number System (http://www.math.wichita.edu/history/topics/num-sys.html**)**
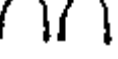
How do we know what the Egyptian language of numbers is? It has been found on the writings on the stones of monument walls of ancient time. Numbers have also been found on pottery, limestone plaques, and on the fragile fibers of the papyrus. The language is composed of hieroglyphs, pictorial signs that represent people, animals, plants, and numbers.

The Egyptians used a written numeration that was changed into hieroglyphic writing, which enabled them to note whole numbers to 1,000,000. It had a decimal base and allowed for the additive principle. In this notation there was a special sign for every power of ten. For I, a vertical line; for 10, a sign with the shape of an upside down U; for 100, a spiral rope; for 1000, a lotus blossom; for 10,000, a raised finger, slightly bent; for 100,000 , a tadpole; and for 1,000,000, a kneeling genie with upraised arms.

Decimal Number	Egyptian Symbol	
1 =		staff
10 =		heel bone
100 =		coil of rope
1000 =		lotus flower
10,000 =		pointing finger
100,000 =		tadpole
1,000,000 =		astonished man

This hieroglyphic numeration was a written version of a concrete counting system using material objects. To represent a number, the sign for each decimal order was repeated as many times as necessary. To make it easier to read the repeated signs they were placed in groups of two, three, or four and arranged vertically.

Example 1.

1 = | 10 = ⌒   100 = 🌀   1000 = 𓆼

2 = || 20 = ⌒⌒   200 = 🌀🌀   2000 = 𓆼𓆼

3 = ||| 30 = ⌒⌒⌒   300 = 🌀🌀🌀   3000 = 𓆼𓆼𓆼

4 = |||| 40 = ⌒⌒/⌒⌒   400 = 🌀🌀/🌀🌀   4000 = 𓆼𓆼/𓆼𓆼

5 = ||/||| 50 = ⌒⌒⌒/⌒⌒   500 = 🌀🌀🌀/🌀🌀   5000 = 𓆼𓆼𓆼/𓆼𓆼

In writing the numbers, the largest decimal order would be written first. The numbers were written from right to left.

Example 2.

46,206 =

Below are some examples from tomb inscriptions.

A	B	C	D
77	700	7000	760,00